October Math Gems

PROBLEM OF THE WEEK 10

§1 Problems

Problem 1.1. Let a, b be non-negative real numbers such that $a + b = 1$. Prove the following inequality is true

$$\frac{a+1}{b+2} + \frac{b+1}{a+2} \leq \frac{4}{3}$$

Solution. We can simplify the inequality and get

$$3a^2 + 3b^2 + 9a + 9b + 12 \leq 4ab + 8a + 8b + 16$$

This is further simplified to

$$3a^2 + 3b^2 \leq 4ab + 3$$

Dividing both sides by 3 and rewriting $a^2 + b^2$ as $1 - 2ab$, we get the following the inequality

$$ab \geq 0$$

This is obviously true. $\square$

Problem 1.2. Let a, b be non-negative real numbers such that $a + b = 2$. Prove the following inequality:

$$\sqrt{a^2 + b + 2} + \sqrt{b^2 + a + 2} \geq 4$$

Solution. Squaring both sides, we get

$$a^2 + b^2 + a + b + 4 + 2\sqrt{(a^2 + b + 2)(b^2 + a + 2)} \geq 16$$

After simplification, we get

$$\sqrt{(a^2 + b + 2)(b^2 + a + 2)} \geq 3 + ab$$

Squaring both sides again, we get

$$(a^2 + b + 2)(b^2 + a + 2) \geq 9 + a^2b^2 + 6ab$$

This is further simplified to

$$1 \geq ab$$

This is true from the given. Hence, proven. $\square$

Problem 1.3. Let $a, b > 0$ and that $a^2 + b^2 = 2$. Prove that

$$\frac{2a^2}{b} + \frac{3b}{a} \geq 5$$

Solution. First, we notice that $ab \leq 1$. Now, we can rewrite the inequality as

$$2a^3 + 6 - 3a \geq 5ab$$

We, then, factorize the LHS to be

$$(2a+1)(a-1)^2 + 5 \geq 5ab$$

This is can be proved to be true by noticing that the LHS will be greater if $a = \frac{-1}{2}$ or $a = 1$, which are the zeros of the polynomial. Any other value is easily noticed to satisfy the expression. $\square$

Problem 1.4. Let a, b be two real numbers. Prove that

$$(a+b)^2 \geq 4ab$$

Solution. First, we start with the inequality $x^2 \geq 0$. Substituting $x = a - b$ and adding $4ab$ to both sides, we get

$$(a+b)^2 \geq 4ab$$

Hence, proven. $\square$

Problem 1.5. Simplify

$$\frac{2^{54}+1}{2^{27}+2^{14}+1}$$

Solution. $\boxed{\textbf{Answer: } 2^{27} - 2^{14} + 1}$

Let $a = 2^{13}$. Now, we have

$$\frac{4a^4+1}{2a^2+2a+1} = \frac{4a^4+4a^2+1-4a^2}{2a^2+2a+1} = \frac{(2a^2+2a+1)(2a^2-2a+1)}{2a^2+2a+1} = 2a^2 - 2a + 1$$

Therefore, this simplifies to $2^{27} - 2^{14} + 1$ $\square$

Problem 1.6. Given that

$$a + \frac{3}{b} = 3$$

$$b + \frac{2}{c} = 2$$

$$c + \frac{1}{a} = 1$$

Find $a + 2b + 3c$

Solution. $\boxed{\textbf{Answer: 2}}$

$$a+\frac{3}{b}-3 = a+\frac{3}{2-\frac{2}{c}}-3 = a+\frac{3}{2-\frac{2}{1-\frac{1}{a}}}-3 = a+\frac{3}{2-\frac{2a}{a-1}}-3 = a-\frac{3(a-1)}{2}-3 = \frac{-3-a}{2}$$

So, $a = -3$, $c = \frac{4}{3}$ and $b = \frac{1}{2}$. Substituting these values, we get

$$a + 2b + 3c = -3 + 1 + 4 = 2$$

$\square$

Problem 1.7. Find the solutions (x, y) to the equations

$$\begin{cases} x^4 + 2x^3 - y = \sqrt{3} - \frac{1}{4} \\ y^4 + 2y^3 - x = -\sqrt{3} - \frac{1}{4} \end{cases}$$

Solution. Summing both equations, we get

$$x^4 + y^4 + 2x^3 + 2y^3 - x - y = -\frac{1}{2}$$

$$\implies (x^4 + 2x^3 - x) + (y^4 + 2y^3 - y) + \frac{1}{2} = 0$$

Noting that

$$(x^2 + x - \frac{1}{2})^2 = x^4 + x^2 + \frac{1}{4} + 2x^3 - x^2 - x = (x^4 + 2x^3 - x) - \frac{1}{4}$$

We rewrite the expression as

$$\implies (x^4 + 2x^3 - x) + (y^4 + 2y^3 - y) + \frac{1}{2} = (x^2 + x - \frac{1}{2})^2 + (y^2 + y - \frac{1}{2})^2 = 0$$

This gives us the solutions

$$(x, y) = \left(\frac{\sqrt{3} - 1}{2}, -\frac{\sqrt{3} + 1}{2} \right)$$

□

Problem 1.8. Determine which number is bigger, 99! or 50^{99}

Solution. $\boxed{\textbf{Answer: } 50^{99}}$

$$\begin{aligned} 99! &= 99 \times 98 \times 97 \times \cdots \times 3 \times 2 \times 1 \\ &= (99 \times 1) \times (98 \times 2) \times (97 \times 3) \times \cdots \times (51 \times 49) \times 50 \\ &= (50^2 - 49^2) \times (50^2 - 48^2) \times (50^2 - 47^2) \times \cdots \times (50^2 - 1^2) \times 50 \\ &< 50^{99} \end{aligned}$$

□

Problem 1.9. Given that

$$x^3 + 3x^2 + 5x - 17 = 0$$

$$y^3 - 3y^2 + 5y + 11 = 0$$

Find $x + y$

Solution. $\boxed{\textbf{Answer: 2}}$
Let $x - 1 = a$ and $y - 1 = b$. Then,

$$a^3 + 2a - 14 = 0$$

$$b^3 + 2b + 14 = 0$$

$$\Rightarrow (a + b)(a^2 - ab + b^2 + 2) = 0$$

Since $a^2 - ab + b^2 + 2 \neq 0$, we get $a + b = 0$. Therefore, $x + y = 2$ □

Problem 1.10. Given that

$$2\cos 40^\circ \sin\theta = sin(160-\theta)$$

Solve for θ.

Solution. $\boxed{\textbf{Answer: } 30^\circ}$

$$2\cos 40 \sin\theta = \sin(20+\theta)$$
$$2\cos 40 \sin\theta = \sin 20\cos\theta + \cos 20\sin\theta$$
$$(2cos40 - cos20)\, sin\theta = sin20cos\theta$$
$$\tan\theta = \frac{\sin 20}{2\cos(60-20) - \cos 20}$$
$$\tan\theta = \frac{\sin 20}{\cos 20 + 2\sin 60\sin 20 - \cos 20} = \frac{\sqrt{3}}{3}$$

Therefore, we conclude that $\theta = 30^\circ$ □

Problem 1.11. The number of positive integral values n for which $(n^3 - 8n^2 + 20n - 13)$ is a prime is ?

Solution.

$$(n^3 - 8n^2 + 20n - 13) = (n-1)(n^2 - 7n + 13)$$

As, $(n^3 - 8n^2 + 20n - 13)$ is a prime number. So, it is in the form $1\times$ itself , there are two possibilities

$$(n-1) = 1 \qquad \text{or} \qquad (n^2 - 7n + 13) = 1$$

For $(n-1) = 1$,

$$n = 1$$

For $(n^2 - 7n + 13) = 1$,

$$n = 3, \quad 4$$

So, there are three possibilities. □

Problem 1.12. What is the largest integer that is a divisor of $(n+1)(n+3)(n+5)(n+7)(n+9)$ for all positive even integer n ?

Solution. As n is an even number. Since **even + odd = odd** . So,

$$(n+1)(n+3)(n+5)(n+7)(n+9)$$

is a product of 5 distinct odd numbers. So, one of these numbers must be divisible by 3 and another one must be divisible by 5.
As, $n = 0$ we get a product of

$$1 \times 3 \times 5 \times 7 \times 9$$

and as 9 doesn't divide the product when $n = 10$. Also, 7 doesn't divide the product when $n = 8$. So, the largest integer that is a divisor of that product is $3 \times 5 = 15$. □

Problem 1.13. For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and $110n^3$. The number $81n^4$ have D positive integer divisors. what is the value of $\frac{D}{5}$?

Solution. If N is a composite number s.t

$$N = a^p \times b^q \times c^r \times \dots$$

where a, b, c are prime numbers. Then,

$$\text{The numbers of divisors } (\tau) = (p+1)(q+1)(r+1)\dots$$

$$110n^3 = 2 \times 5 \times 11 \times n^3$$

As the number n has 110 positive integer divisors and $110 = 2 \times 5 \times 11$. We can see that $n = 2^3 \times 5$

$$110n^3 = 110 \times (2^3 \times 5)^3 = 2 \times 5 \times 11 \times 2^9 \times 5^3 = 2^{10} \times 5^4 \times 11$$

As $110n^3 = 2^{10} \times 5^4 \times 11$, the number of divisors are $(10+1)(4+1)(1+1) = 11 \times 5 \times 2 = 110$.
Hence we are correct.
The number $81n^4$ is about

$$81(2^3 \times 5)^4 = 3^4 \times 2^{12} \times 5^4$$

So, the number of divisors D is

$$(4+1)(12+1)(4+1) = 325 \implies \frac{D}{5} = \frac{325}{5} = 65$$

□

Problem 1.14. Given that

$$x = \lfloor \sqrt[3]{1} \rfloor + \lfloor \sqrt[3]{2} \rfloor + \lfloor \sqrt[3]{3} \rfloor + \lfloor \sqrt[3]{4} \rfloor + \lfloor \sqrt[3]{5} \rfloor + \cdots + \lfloor \sqrt[3]{7999} \rfloor$$

find the value of $\lfloor \frac{x}{5000} \rfloor$, where $\lfloor y \rfloor$ denotes to the greatest integer function less than or equal to y.

Solution. The numbers that have a cubic root from 1 to 8000 is

$$1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728, 2197, 2744, 3375, 4096, 4913, 5832, 6859, 8000$$

We can see that $\lfloor \sqrt[3]{1} \rfloor = 1$
$\lfloor \sqrt[3]{2} \rfloor = 1$
$\lfloor \sqrt[3]{3} \rfloor = 1$
$\lfloor \sqrt[3]{4} \rfloor = 1$
$\lfloor \sqrt[3]{5} \rfloor = 1$
$\lfloor \sqrt[3]{6} \rfloor = 1$
$\lfloor \sqrt[3]{7} \rfloor = 1$
The number 1 is repeated 7 times which are $(8-1)$. So,

$$x = (8-1) \times 1 + (27-8) \times 2 + (64-27) \times 3 + (125-64) \times 4 + (216-125) \times 5 + \cdots + (8000-6859) \times 19 =$$

$$115900$$

So, $\lfloor \frac{115900}{5000} \rfloor = 23$ □

Problem 1.15. How many digits has the number $9^{30}4^{71}$?

Solution.

$$j = 1 + \lfloor \log_{10} n \rfloor \quad \text{(Where } j \text{ is the number of digits, and } n \text{ is the number)}$$
$$\text{Number of digits of } 9^{30}4^{71} \ = 1 + \lfloor \log_{10} 9^{30}4^{71} \rfloor = 1 + 71 = 72$$

□

Problem 1.16. If x is a real number that satisfies

$$\lfloor x + \frac{11}{100} \rfloor + \lfloor x + \frac{12}{100} \rfloor + \lfloor x + \frac{13}{100} \rfloor + \lfloor x + \frac{14}{100} \rfloor + \cdots + \lfloor x + \frac{99}{100} \rfloor = 765$$

find the value of $900 - \lfloor 100x \rfloor$.

Solution. Suppose that $x = a + b$ where a is the integer part and b is the fraction or decimal part. So, we can write the equation as

$$\lfloor a + b + \frac{11}{100} \rfloor + \lfloor a + b + \frac{12}{100} \rfloor + \lfloor a + b + \frac{13}{100} \rfloor + \lfloor a + b + \frac{14}{100} \rfloor + \cdots + \lfloor a + b + \frac{99}{100} \rfloor = 765$$

We can write $\lfloor a + b \rfloor = a + \lfloor b \rfloor$ as a is an integer part and b is the fraction part. So, we can write the equation as

$$a + a + \cdots + a \quad \text{(89 times)} \ + \lfloor b + \frac{11}{100} \rfloor + \lfloor b + \frac{12}{100} \rfloor + \lfloor b + \frac{13}{100} \rfloor + \lfloor b + \frac{14}{100} \rfloor + \cdots + \lfloor b + \frac{99}{100} \rfloor = 765$$
$$89a \ + \lfloor b + \frac{11}{100} \rfloor + \lfloor b + \frac{12}{100} \rfloor + \lfloor b + \frac{13}{100} \rfloor + \lfloor b + \frac{14}{100} \rfloor + \cdots + \lfloor b + \frac{99}{100} \rfloor = 765$$

As the fraction in the greatest integer function for example $\frac{11}{100}$ is less than 1 and $0 \leq b < 1$ so the maximum value of

$$\lfloor b + \frac{\text{numbers from 11 to 99}}{100} \rfloor$$

will be one. So, the maximum value of

$$\lfloor b + \frac{11}{100} \rfloor + \lfloor b + \frac{12}{100} \rfloor + \lfloor b + \frac{13}{100} \rfloor + \lfloor b + \frac{14}{100} \rfloor + \cdots + \lfloor b + \frac{99}{100} \rfloor = 89$$

Let's get the value of a, suppose that $a = 8$,

$$(89 \times 8) + \lfloor b + \frac{11}{100} \rfloor + \lfloor b + \frac{12}{100} \rfloor + \lfloor b + \frac{13}{100} \rfloor + \lfloor b + \frac{14}{100} \rfloor + \cdots + \lfloor b + \frac{99}{100} \rfloor = 765$$
$$\lfloor b + \frac{11}{100} \rfloor + \lfloor b + \frac{12}{100} \rfloor + \lfloor b + \frac{13}{100} \rfloor + \lfloor b + \frac{14}{100} \rfloor + \cdots + \lfloor b + \frac{99}{100} \rfloor = 53$$

and it is correct as we mentioned before that

$$\lfloor b + \frac{11}{100} \rfloor + \lfloor b + \frac{12}{100} \rfloor + \lfloor b + \frac{13}{100} \rfloor + \lfloor b + \frac{14}{100} \rfloor + \cdots + \lfloor b + \frac{99}{100} \rfloor$$

has a maximum value 89 and 53 is less than 89. Now, we want to know the 53^{th} term from the ending because it will be equal to one. So,

$$b + 0.47 = 1 \implies b = 0.53$$
$$x = a + b = 8 + 0.53 = 8.53$$
$$900 - \lfloor 100x \rfloor = 900 - 853 = 47$$

□

Problem 1.17. For any real number x, let $\lceil x \rceil$ denote the smallest integer that is greater than or equal to x and $\lfloor x \rfloor$ denotes to the greatest integer function less than or equal to x. Find the value of

$$2010 - \sum_{k=1}^{2010} \lceil \frac{2010}{k} - \lfloor \frac{2100}{k} \rfloor \rceil$$

Solution. First, we know that $\lceil a - \lfloor a \rfloor \rceil = 0$, where a is an integer. Also, we know that if the number is divided by any of its divisors gives us an integer number (not decimal). The number of divisors in 2010 is 16. As

$$2010 = 2^1 \times 3^1 \times 7^1 \times 67^1$$

$$\text{the number of divisors of } 2010 = (1+1)(1+1)(1+1)(1+1) = 2^4 = 16$$

Second, we know that if $\frac{a}{b}$ and $a < b$ it will give us a number less than one.

$$\lceil m \rceil \text{ for } m < 1 \text{ is equal to } 1$$

$$\sum_{k=1}^{2010} \lceil \frac{2010}{k} - \lfloor \frac{2100}{k} \rfloor \rceil = 1 \times (2010 - 16) = 1994$$

For example (for more explanation):

$$\lceil \frac{2010}{2} - \lfloor \frac{2100}{2} \rfloor \rceil = 0$$

As 2 is one of the divisors of 2010.

$$\lceil \frac{2010}{11} - \lfloor \frac{2100}{11} \rfloor \rceil = \lceil \frac{8}{11} \rceil = 1$$

As $8 < 11$.

$$2010 - \sum_{k=1}^{2010} \lceil \frac{2010}{k} - \lfloor \frac{2100}{k} \rfloor \rceil = 2010 - 1994 = 16 \qquad \text{(which is the number of divisors of 2010)}$$

□

Problem 1.18. Given $x + y = \sqrt{3\sqrt{5} - \sqrt{2}}$ and $x - y = \sqrt{3\sqrt{2} - \sqrt{5}}$. What is the value of xy?

Solution.

$$(x+y)^2 = x^2 + 2xy + y^2 = 3\sqrt{5} - \sqrt{2} \to (1)$$
$$(x-y)^2 = x^2 - 2xy + y^2 = 3\sqrt{2} - \sqrt{5} \to (2)$$

Now, we can solve for xy, Multiply the equation (2) by -1 and add with equation (1).

$$4xy = 3\sqrt{5} - \sqrt{2} - 3\sqrt{2} + \sqrt{5} = 3(\sqrt{5} - \sqrt{2}) + (\sqrt{5} - \sqrt{2} = 4(\sqrt{5} - \sqrt{2}) \implies xy = \sqrt{5} - \sqrt{2}$$

□

Problem 1.19. Evaluate x in the simplest form then find the sum of all digits of x. Where x is given as

$$x = \sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{\cdots}}}}}$$

Solution. We can write x as

$$x = \sqrt{2008 + 2007x} \implies x^2 = 2008 + 2007x$$

Now, it will be easier to solve for x.

$$x^2 - 2007x - 2008 = 0 \implies x = 2008, \quad x = -1 \quad \text{(refused)}$$

So, the solution is 2008 and the sum of its digits is $2 + 8 + 0 + 0 = 10$ $\square$

Problem 1.20. Find the number of ordered pairs of positive integers (x, y) that satisfy the equation

$$x\sqrt{y} + y\sqrt{x} + \sqrt{2009xy} - \sqrt{2009x} - \sqrt{2009y} - 2009 = 0$$

Solution.

$$(\sqrt{x} + \sqrt{y} + \sqrt{2009})(\sqrt{xy} - \sqrt{2009}) = 0 \implies (\sqrt{xy} - \sqrt{2009}) = 0$$

since

$$(\sqrt{x} + \sqrt{y} + \sqrt{2009}) > 0$$

$$(\sqrt{xy} - \sqrt{2009}) = 0 \implies xy = 2009$$

Hence,

$$2009 = 7^2 \times 41$$

So the number of ordered pairs (x, y) is $(2 + 1) \times (1 + 1) = 3 \times 2 = 6$ $\square$
